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On Stability and Bifurcation of Solutions of the Belousouv–Zhabotinskii Reaction System

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In this paper the three-dimensional Belousouv–Zhabotinskii system is considered. The Hopf bifurcation and center manifold theories are applied. The stability of periodic solutions is established. © 1990 Academic Press, Inc.

1. INTRODUCTION

The Belousouv–Zhabotinskii (B–Z) reaction is one of the most interesting of chemical reactions for which the chemical mechanism has received a great deal of attention by a number of authors (see [5, 11, 14, 15, 17, and references therein]). The detailed B–Z model leads to differential equations that are much too complex for qualitative mathematical analysis. In 1974 Field and Noyes [3] reduced the mechanism to five essential steps, which appears to retain the important features of the complete system (see Tyson [16, p. 280]). Their model, called the oregonator, is

$$(\mathrm{R}_1) \quad A+Y\to X+P$$
$$(\mathrm{R}_2) \quad X+Y\to 2P$$
$$(\mathrm{R}_3) \quad A+X\to 2X+2Z$$
$$(\mathrm{R}_4) \quad 2X\to A+P$$
$$(\mathrm{R}_5) \quad B+Z\to hy,$$

where $A=\mathrm{BrO}_3^-$, $B=\mathrm{BrMA}$, $P=\mathrm{HOBr}$, $X=\mathrm{HBrO}_2$, $Y=\mathrm{Br}^-$, $Z=\mathrm{Ce}^{4+}$, and $2h$ is the stoichiometric factor appearing in (R_5).

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They describe the dynamics of the model by the differential equations

$$\begin{aligned}\frac{dx}{dT}&=K_1AY-K_2XY+K_3AX-2K_4X^2\\ \frac{dy}{dT}&=-K_1AY-K_2XY+hK_5BZ\\ \frac{dz}{dT}&=2K_3AX-K_5BZ,\end{aligned}\tag{1.1}$$

where X, Y, Z are concentrations of the intermediates $HBrO_2$ (bromous acid), Br^- (bromide ion), and Ce^{4+} (cerium) in moles liter^{-1} and T is time in seconds. The K_i's denote the specific forward rate constants for reactions (R_1)–(R_5).

In 1979, Geisler and Follner (Ref. [5, p. 111, Eqs. (14)–(17)]) described the reaction by a more convenient model from an experimental and theoretical point of view (see [16, p. 286]).

They discussed their observation in terms of Field–Noyes equations (1.1) with $B=0$ and additional terms representing material flux in a continuous flow, stirred-tank reactor. The Geisler and Follner model is

$$\begin{aligned}\frac{dx}{dT}&=K_1AY-K_2XY+K_3AX-2K_4X^2-K_RX\\ \frac{dy}{dT}&=-K_1AY-K_2XY+K_R(Y^0-Y)\\ \frac{dz}{dT}&=-2K_3AX+K_R(Z^0-Z).\end{aligned}\tag{1.2}$$

Here $Z=[Ce^{3+}]$, $K_R=T_R^{-1}$, $T_R=$ residence time of the reactor, and Y^0, Z^0 are concentrations of Br^{1-} and Ce^{3+} in the feed stream.

Using the substitution

$$x=\frac{2k_4}{k_3A}X,\qquad y=\frac{k_2}{k_3A}Y,\qquad t=k_3AT$$

$$\mu=\frac{2k_1k_4}{k_2k_3},\qquad k=\frac{k_R}{k_3A},\qquad \alpha=\frac{1}{\varepsilon}=\frac{k_2}{2k_4}.$$

Tyson [16, pp. 286–287, Eqs. (21)–(24)] reduced (1.2) to the following 2-component system:

$$\begin{aligned}\dot{x}&=(1-k)x+\mu y-x^2-xy=F_1(x, y),\\ \dot{y}&=ky^0-(\alpha\mu+k)\,y-\alpha xy=F_2(x, y).\end{aligned}\tag{1.3}$$

Using the limits $\varepsilon \to 0$, $\mu \to 0$, he has studied the multiplicity of the steady states of (1.2) by reducing (1.3) to a single equation.

This paper is the second of two papers on the solutions of the B–Z system. In the first paper, multiplicity, indices, bifurcation, and the stability of equilibria of the B–Z system are discussed using (1.3). We may note that the bifurcation of solutions of the B–Z system discussed in [11, pp. 209–211, Eqs. (1)–(3)] depends on another reduction form of (1.1) using a singular perturbation technique. In this paper, we study the 3-component system given by [5, p. 110, Eqs. (7)–(9)] with the natural parameters without reduction. In Section 2, we summarize most of the bifurcation results given in [2]. Then, we give sufficient conditions for asymptotic stability and for supercritical or subcritical Hopf bifurcation by a criteria as given in [9, p. 203]. Moreover, the center manifold theory is discussed in such a manner as in [1, p. 8; 5, p. 52]. In Section 3, we discuss the 3-component system (1.2). Then, using a technique given by Pimbley [12, pp. 105–107; 13, p. 172], we apply Hopf bifurcation theory [10, p. 165] and center manifold theory [7, p. 52].

2. Two-Dimensional B–Z Model

The critical points $(\bar{x}, \bar{y})$ are solutions of the simultaneous equations

$$\begin{aligned} F_1(\bar{x}, \bar{y}) &= 0, \\ F_2(\bar{x}, \bar{y}) &= 0, \end{aligned} \tag{2.1}$$

where F_1 and F_2 are given by (1.3).

Now, if $\bar{y}$ is eliminated from (2.1), then the result is the third-order equation

$$\bar{x}^3 - c_1\bar{x}^2 - c_2\bar{x} - c_3 = 0, \tag{2.2}$$

where

$$c_1 = 1 - k - k/\alpha - \mu,$$

$$c_2 = (1-k)\left(\mu + \frac{k}{\alpha}\right) - k\frac{y^0}{\alpha},$$

and

$$c_3 = ky^0/\alpha.$$

As stated in [2, p. 2], we assume that $k\varepsilon(0, 1)$, μ is sufficiently small, and α is large enough such that $c_i > 0$, $i = 1, 2, 3$.

Using the monotonicity of y^0 with $\bar{x}$ when α, k, μ are permitted to be fixed, we have

$$\alpha k(\bar{x}-\mu)^2 \frac{dy^0}{d\bar{x}} = -2\bar{x}^3 + (c_1+3\mu)\,\bar{x}^2 - 2\mu c_1 \bar{x} - (\mu c_2 - c_3) = H(\bar{x}), \tag{2.3}$$

say. It was shown [2, Lemma 1] using (2.3) that the relation between y^0 and $\bar{x}$ may look like Fig. 1.

It is clear (see Fig. 1.) that there is no positive steady state for (1.3), where $\bar{x} < \mu$, while for $\bar{x} > \mu$ the uniqueness and mutliplicity of steady states depend on the range of y^0.

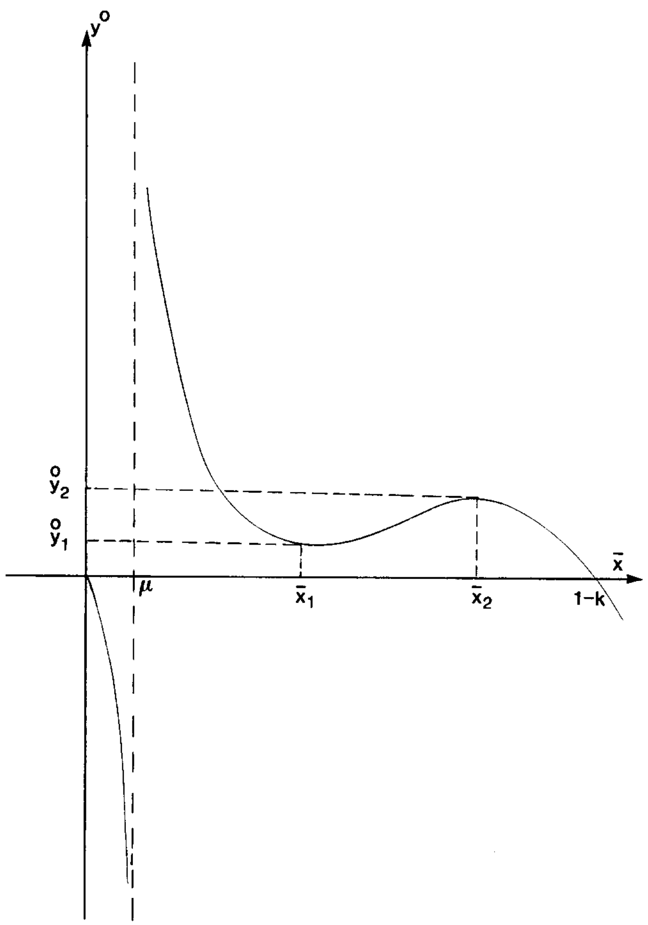

FIGURE 1

It was also shown [2, Eq. (3.4)] that

$$\Delta = \det J = -k(\bar{x} - \mu)\frac{dy^0}{d\bar{x}}, \tag{2.4}$$

where J is the Jacobian matrix

$$J = \begin{bmatrix} \left(1 - k - 2\bar{x} - \dfrac{ky^0}{[\alpha(\mu + \bar{x}) + k]}\right) & -x(\bar{x} - \mu) \\ -\dfrac{\alpha k y^0}{[\alpha(\mu + \bar{x}) + k]} & -[\alpha(\mu + \bar{x}) + k \end{bmatrix} \tag{2.5}$$

The bifurcation loci which divide regions where there is a single critical point from regions where there are three critical points are given by

$$F_1 = F_2 = \Delta = 0. \tag{2.6}$$

From (2.3) and (2.4), the bifurcation equation (2.6) is equivalent to the equation

$$H(\bar{x}) = 0. \tag{2.7}$$

y^0 being chosen as a bifurcation parameter, a branch of stationary solutions is obtained, say, $\bar{x}(y^0)$. This branch occurs at y_1^0 and y_2^0 in Fig. 1.

Now, since by (2.5), for sufficiently large α and for $k \in (0, 1)$, $\bar{x} \in (\mu, 1 - k)$, we see that

$$\sigma = \operatorname{tr} J < 0, \tag{2.8}$$

the necessary and sufficient condition for an asymptotically stable critical point is

$$\Delta < 0.$$

Thus, by (2.4), the critical point is asymptotically stable if and only if $dy^0/d\bar{x} > 0$. (This is clearly satisfied for $x \in (\bar{x}_1, \bar{x}_2)$ in Fig. 1.)

Now, assume that there exists a value y_c^0 of y^0 such that $\sigma = 0$ and $\Delta > 0$. Then,

$$y_c^0 = \frac{[1 - \alpha\mu - 2k - (\alpha + 2)\bar{x}][\alpha(\mu + \bar{x}) + k]}{k}. \tag{2.9}$$

Consequently, there exist two pure imaginary eigenvalues of J_0 (the Jacobian matrix J at the critical $(\bar{x}, \bar{y})$), say

$$\lambda_1 = \bar{\lambda}_2 = \gamma + i\omega \qquad (\omega > 0), \tag{2.10}$$

where $\gamma = \frac{1}{2}$ trace J_0 and (2.11)

$$\omega_0^2 = \det J_0. \tag{2.12}$$

If $y^0 = y_c^0$, then

$$\lambda_1 = \bar{\lambda}_2 = i\omega_0 \qquad (\omega_0 > 0). \tag{2.13}$$

Thus, it is well known [7, pp. 78–84], by the implicit function theorem, that there is a smooth curve of steady states $(\bar{x}(y^0), \bar{y}(y^0), y^0)$ in the neighborhood of $(\bar{x}_c, \bar{y}_c, y_c^0)$, where

$$\bar{x}_c = \bar{x}(y_c^0),$$
$$\bar{y}_c = \bar{y}(y_c^0).$$

The eigenvalues $\lambda_1(y^0)$ and $\lambda_2(y^0)$ vary smoothly with y^0 and $\lambda_1(y_c^0) = \bar{\lambda}_2(y_c^0) = i\omega_0$. Moreover, since by (2.5),

$$\text{trace} \left.\frac{d}{dy^0}(J)\right|_{y^0 = y_c^0} \neq 0,$$

then the Hopf bifurcation theorem [10, p. 165] is satisfied and we have:

LEMMA 2.1. *At $y^0 = y_c^0$, there exists a one-parameter family of periodic solutions bifurcating from the critical point $(\bar{x}, \bar{y})$ which is asymptotically orbitally stable (limit cycle).*

Proof. Since there exist two pure imaginary eigenvalues $\pm i\omega_0$ at $y^0 = y_c^0$ and

$$\text{trace}\left(\left.\frac{d}{dy^0}(J)\right|_{y^0 = y_c^0}\right) \neq 0,$$

by Friedrichs' theorem [4, p. 94], there exists a one-parameter family of periodic solutions bifurcating from $(\bar{x}, \bar{y})$.

Now, by (1.3) and (2.1),

$$\nabla \cdot F = -\left[2k + 2\bar{x} + \frac{ky^0}{\alpha(\mu + \bar{x}) + k} + \alpha(\mu + \bar{x}) - 1\right] < 0$$

for sufficiently large α. Using the Poincaré criterion (see [8, p. 238])

$$\nabla \cdot F\, d\tau = -\int_0^{t_0} \left[2k + 2\bar{x} + \frac{ky^0}{\alpha(\mu + \bar{x}) + k} + \alpha(\mu + \bar{x}) - 1\right] d\tau < 0.$$

This completes the proof.

There are well-known criteria to show that the limit cycle exists and is stable for $y^0 < y_c^0$ (supercritical bifurcation), or that the limit cycle exists and is unstable for $y^0 > y_c^0$ (subcritical Hopf bifurcation). See [10, pp. 16–20;

6, p. 152]. Here, we use the improved criterion given in [9, p. 203] (see Appendix I).

In fact, the bifurcation is supercritical (or subcritical) according to $A<0$ (or $A>0$), where A is given by

$$A=\frac{\gamma}{\omega_0^2 \alpha k y_c^0}\left[2\left(\frac{\alpha k y^0}{\gamma}-2\gamma\right)(\gamma+\bar{x}-\mu)+2\alpha\gamma(\bar{x}-\mu)(\alpha+1)\right]. \quad (2.14)$$

Now, to apply the center manifold theorem [7, p. 52; 6, p. 152] we write the system (1.3) in the form

$$\dot{y}=Dy+\psi(y, v), \quad (2.15)$$

where

$$D=T^{-1}JT=\begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$$

(see Appendix II), $v=y^0-y_c^0$, and $\psi(y, v)=T^{-1}F(Ty+\bar{x}, Y^0-v)$. Thus, it is clear that the isolated stationary point for (2.15) is the origin, and the critical value of the bifurcation parameter is $v=0$ with

$$\lambda(v)\equiv\gamma(v)+i\omega(v)=\lambda_i(y_c^0+v).$$

To prove the existence of a center manifold for the new system (2.15) we use the well-known technique [1, p. 8; 7, p. 52] to write (2.15) as

$$\dot{y}=\phi(y, v),$$
$$\dot{v}=0,$$

where

$$\phi(y, v)=Dy+\psi(y, v).$$

By [7, p. 52; 6, p. 152], there exists at $(y, v)=(0, 0)\in R^2\times R^1$, a three-dimension center manifold C which is locally invariant, and locally attracting.

Moreover, C contains the origin and an interval $\{|v|<v_0\}$ of the v-axis (assuming $\phi(0, v)=0$ for $|v|<v_0$).

3. Geisler and Follner B–Z Model

Let $(\bar{x}, \bar{y}, \bar{z})$ denote critical points of the 3-component system (1.2) derived by Geisler and Follner [5, p. 111, Eqs. (14)–(17)] for the B–Z reaction [16, p. 286, Eq. (.1)]. Then,

$$
\begin{aligned}
(k_3A-k_R)\bar{x}+k_1A\bar{y}-2k_4\bar{x}^2-k_2\bar{x}\bar{y}&=0\\
k_R\, y^0-(k_1A+k_R)\,\bar{y}-k_2\bar{x}\bar{y}&=0 \qquad (3.1)\\
k_R z^0-2k_3A\bar{x}-k_R\bar{z}&=0.
\end{aligned}
$$

Using the substitutions

$$x=\bar{x}+\varepsilon y_1, \qquad y=\bar{y}+\varepsilon y_2, \qquad z=\bar{z}+\varepsilon y_3,$$

Eqs. (3.1) take the form

$$\dot{y}=Jy+\varepsilon B(y), \tag{3.2}$$

where J is the coefficient matrix

$$J=\begin{bmatrix} k_3A-k_R-4k_4\bar{x}-k_2\,\bar{y} & k_1A-k_2\bar{x} & 0\\ -k_2\,\bar{y} & -[k_1A+k_R+k_2\bar{x}] & 0\\ -2k_3A & 0 & -k_R \end{bmatrix} \tag{3.3}$$

and the nonlinear part is given by

$$B(y)=\begin{bmatrix} -2k_4\,y_1^2-k_2\,y_1\,y_2\\ -k_2\,y_1\,y_2\\ 0 \end{bmatrix} \tag{3.4}$$

Now, the characteristic roots of J obey the characteristic equation

$$\lambda^3-\sigma\lambda^2+\hat{\sigma}\lambda-\Delta=0, \tag{3.5}$$

where

$$
\begin{aligned}
\sigma&=\operatorname{trace} J\\
&=k_3A-3k_R-4k_4\bar{x}-k_2\,\bar{y}-k_1A-k_2\bar{x},\\
\hat{\sigma}&=\text{trâce}\, J=F_{2y}(F_{1x}+F_{3z})-F_{2x}F_{1y}-F_{1z}F_{3x}\\
&=(k_1A+k_R+k_2\bar{x})(2k_R+4k_4\bar{x}+k_2\,\bar{y}-k_3A)+k_2\,\bar{y}(k_1A-k_2\bar{x})\\
&\quad+k_R(k_R+4k_4\bar{x}+k_2\,\bar{y}-k_3A)\\
&=(k_1A+k_R)(2k_R+4k_4\bar{x}+k_2\,\bar{y}-k_3A)+k_2\bar{x}(2k_R+4k_4\bar{x}-k_3A)\\
&\quad+k_2\,\bar{y}\cdot k_1A+k_R(k_R+4k_4\bar{x}+k_2\,\bar{y}-k_3A)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta=\det J=&-k_R[(k_2\,\bar{y}+k_R+4k_4\bar{x}-k_3A)(k_1A+k_R+k_2\bar{x})\\
&+k_2\,\bar{y}(k_2\bar{x}-k_1A)].
\end{aligned}\tag{3.6}
$$

Remark. Assume that $k_2\bar{y} > k_3A - 4k_4\bar{x} > 0$ and $k_2\bar{x} > k_1A$. Then, by (3.6), it is clear that

$$\sigma < 0, \qquad \hat{\sigma} > 0, \qquad \Delta < 0, \qquad \frac{d}{dk_{\mathrm{R}}}\sigma < 0.$$

Now, we choose k_{R} to be the bifurcation parameter for the system (3.1). Let k_{R}^* be the value of k_{R} at which the characteristic equation (3.5) has two pure imaginary roots $\lambda_{1,2}$. Since there exists at least one real root of the cubic equation (3.5), λ_3 say, we have the factorization

$$(\lambda - \lambda_3)[\lambda^2 + (\lambda_3 - \sigma) + (\lambda_3^2 - \sigma\lambda_3 + \hat{\sigma})] = 0. \tag{3.7}$$

Since by (3.5),

$$\lambda_1 + \lambda_2 + \lambda_3 = \sigma,$$

at $k_{\mathrm{R}} = k_{\mathrm{R}}^*$, we get

$$\lambda_3 = \sigma, \qquad \lambda_1 = \bar{\lambda}_2,$$
$$\lambda_{1,2} = -\tfrac{1}{2}\{(\lambda_3 - \sigma) \pm \sqrt{(\lambda_3 - \sigma)^2 - 4(\lambda_3^2 - \sigma\lambda_3 + \hat{\sigma})}\}. \tag{3.8}$$

Now, at $k_{\mathrm{R}} = k_{\mathrm{R}}^*$, Eq. (3.5) can be written as

$$F_{k_{\mathrm{R}}}(\sigma) = \hat{\sigma}\sigma - \Delta = 0. \tag{3.9}$$

Hence, since $\Delta < 0$ and $\hat{\sigma} > 0$, at $k = k_{\mathrm{R}}^*$ we should have $\lambda_3 = \sigma < 0$. Also, the critical value $k_{\mathrm{R}} = k_{\mathrm{R}}^* > 0$ is a solution of (3.9) which can be seen by (3.6) to be the cubic equation in k_{R}

$$-8k_{\mathrm{R}}^3 - \delta_1 k_{\mathrm{R}}^2 - \delta_2 k_{\mathrm{R}} + \delta_3 = 0, \tag{3.10}$$

where

$$\delta_1 = 8(k_1A + k_2\bar{x} + 4k_4\bar{x} + k_2\bar{y} - k_3A),$$
$$\delta_2 = (k_2\bar{x} + k_1A + k_2\bar{y} + 4k_4\bar{x} - k_3A)[2k_2\bar{x} + 2k_1A + k_2\bar{y} + 4k_4\bar{x} - k_3A]$$
$$+ 3k_1A(4k_4\bar{x} + 7/3k_2\bar{y} - k_3A) + 3k_2\bar{x}(4k_4\bar{x} - k_3A - 1/3k\bar{y})$$
$$+ (4k\bar{x} + k_2\bar{y} - k_3A)^2,$$

and

$$\delta_3 = (k_3A - 4k_4\bar{x} - k_2\bar{y} - k_1A - k_2\bar{x})[k_1A(4k_4\bar{x} + k_2\bar{y} - k_3A)$$
$$+ k_2\bar{x}(4k_4\bar{x} - k_3A) + k_2\bar{y}k_1A].$$

Conversely, knowing that $\Delta < 0$ and $\sigma < 0$, $k_R > 0$, if we can solve Eq. (3.10) for $k_R^* > 0$, we then know that $\hat{\sigma} > 0$, $\lambda_3 = \sigma$, and $\lambda_{1,2}$ are conjugate imaginary.

Since by (3.10), $\delta_1 > 0$, $\delta_2 > 0$, and $\delta_3 > 0$,

$$F_0(\sigma) = \delta_3 > 0, \qquad \lim_{k_R \to \infty} F_{k_R}(\sigma) = -\infty, \qquad \lim_{k_R \to -\infty} F_{k_R}(\sigma) = \infty.$$

Thus, k_R^* is uniquely determined (see Fig. 2). Suppose $k_R > k_R^*$. Then, from Eq. (3.10) we see that $F_{k_R}(\sigma) < 0$. On the other hand, if $k_R < k_R^*$ then $F_{k_R}(\sigma) > 0$ (see Fig. 3).

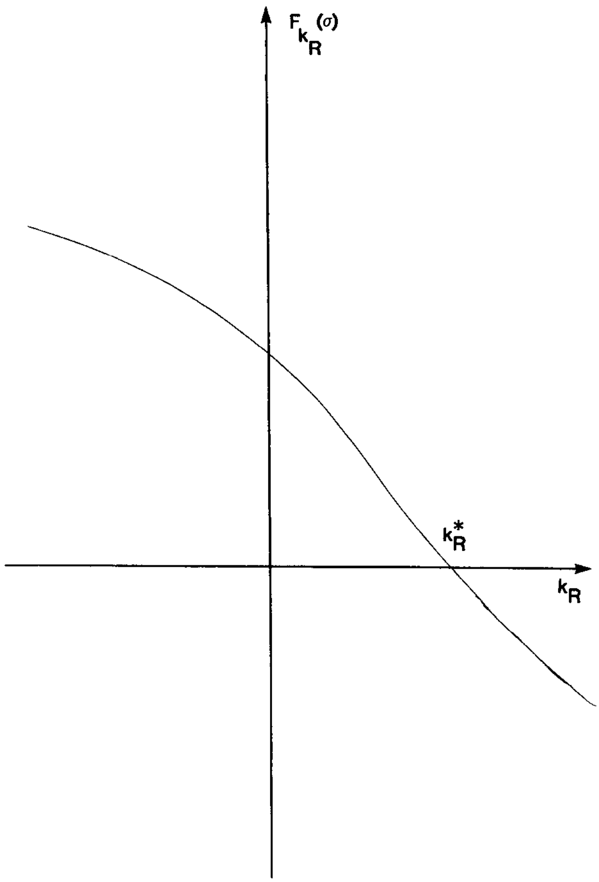

FIGURE 2

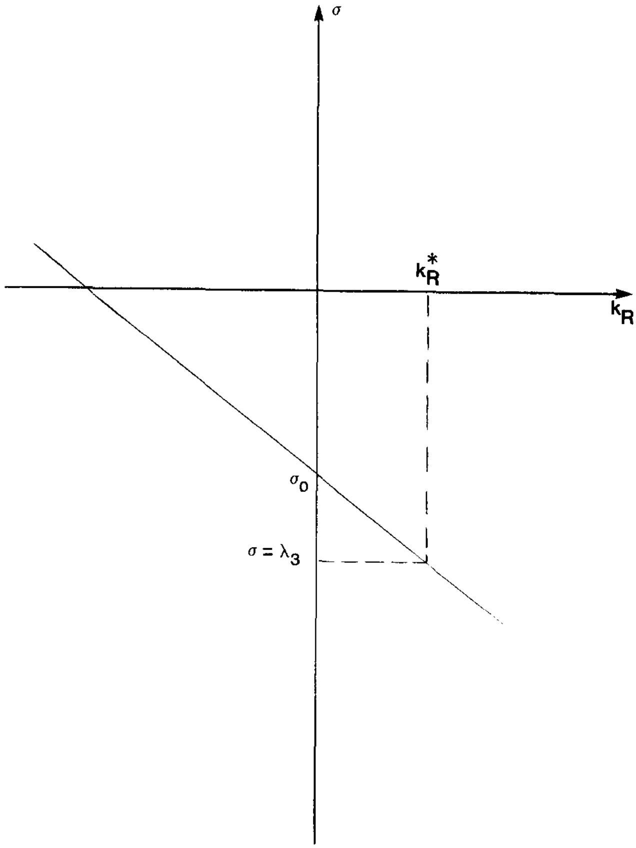

FIGURE 3

Now, since by Eq. (3.5) $\lambda_3 < 0$ and

$$\begin{aligned} F_{k_R}(\sigma) &= \sigma\hat{\sigma} - \Delta \\ &= (\sigma - \lambda_3)(\lambda_1\lambda_2 + \sigma\lambda_3), \\ \operatorname{sgn} F_{k_R}(\sigma) &= \operatorname{sgn}(\sigma - \lambda_3). \end{aligned}$$

Consequently, if $k_R > k_R^*$, then $R_e\lambda_{1,2} = \frac{1}{2}(\sigma - \lambda_3) < 0$ and for $k_R < k_R^*$, $R_e\lambda_{1,2} > 0$ (see Fig. 3).

Remark. If $k_R > k_R^*$ is yet small enough that $\lambda_{1,2}$ are complex conjugates, then $(\bar{x}, \bar{y}, \bar{z})$ is a focus. If $k_R < k_R^*$ is yet large enough that $\lambda_{1,2}$ are complex conjugates, then $(\bar{x}, \bar{y}, \bar{z})$ is a saddle focus (see [12, p. 160])

THEOREM 3.1. *There exists a one-parameter family of periodic solutions bifurcating from* $(\bar{x}, \bar{y}, \bar{z})$ *at* $k_R = k_R^*$ *with period* T, *where* $T \to T_0$ *as* $k_R \to k_R^*$ *and where* $T_0 = 2\pi/\omega_0 = 2\pi/\sqrt{\hat{\sigma}}$ *and* $\hat{\sigma}$ *as given in* (3.6).

Proof. By the above discussion, we see that as k_R is increased through k_R^*, there exists a pair of complex conjugate eigenvalues $\lambda_{1,2}$ of the Jacobian matrix J.

Since at $k_R = k_R^*$, $\lambda_3 = \sigma$, $\lambda_{1,2} = \pm i\sqrt{\hat{\sigma}} = \pm i\omega_0$, where it is clear that $\omega_0 > 0$.

Now, since for $\lambda_1 = \bar{\lambda}_2$,

$$\operatorname{Re} \lambda_2 = \tfrac{1}{2}(\lambda_2 + \bar{\lambda}_2) = 0 \qquad \text{at} \quad k_R = k_R^*,$$

and by the above discussion we see that

$$\begin{aligned} \operatorname{Re} \lambda_2 &> 0 \qquad \text{for} \quad k_R < k_R^* \\ &< 0 \qquad \text{for} \quad k_R > k_R^*. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dk_R} (\operatorname{Re} \lambda_2)|_{k_R = k_R^*} &= -\tfrac{1}{2} \frac{d}{dk_R} (\lambda_3 - \sigma)|_{k_R = k_R^*} \\ &= \operatorname{Re} \left(\frac{d}{dk_R} \lambda_2 \right)\bigg|_{k_R = k_R^*} < 0. \end{aligned}$$

This completes the proof.

Remark. In a similar fashion as seen in [13, p. 172, line 2], the Jacobian matrix J can be diagonalized to be

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & \gamma_{22} & 0 \\ 0 & 0 & \hat{\sigma} \end{bmatrix},$$

where $\gamma_{12} = -\gamma_{21} = -\sqrt{\hat{\sigma}}$.

Thus, putting $\rho = k_R - k_R^*$, at $\rho = 0$, $\gamma_{11} = \gamma_{22} = 0$, we can write (1.1) in the form

$$\begin{aligned} \dot{y} &= \phi(y, \rho) \\ \dot{\rho} &= 0. \end{aligned}$$

Then, a center manifold C exists for this canonical suspended system at $(y, \rho) = (0, 0) \in R^3 \times R^1$ (see [7, p. 52]).

APPENDIXES

(I)

A criterion for direction of bifurcation (see [9, p. 203, A16]):
Consider the two-dimensional system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} F(x, y, y^0) \\ G(x, y, y^0) \end{bmatrix}.$$

Suppose that at $y^0 = y_c^0$ there exists a Hopf bifurcation. Then, the bifurcation is supercritical (or subcritical) if $A < 0$ (or $A > 0$). Here the quantity A is given by

$$\begin{aligned} A = \frac{1}{\omega^2 G_x} \{ & -F_y(F_{xxx} + G_{xxy}) + 2F_x(F_{xxy} + G_{xyy}) + G_x(F_{xyy} + G_{yyy})\, \omega^2 \\ & + (F_x F_{xx} + G_x F_{xy})(-F_y F_{xx} + 2F_x F_{xy} + G_x F_{yy}) \\ & - (F_x G_{yy} - F_y G_{xy})(-F_y G_{xx} + 2F_x G_{xy} + G_x G_{yy}) \\ & - F_y^2 F_{xx} G_{xx} + F_x F_y (F_{xy} G_{xx} + F_{xx} G_{xy}) \\ & + G_x^2 F_{yy} G_{yy} + F_x G_x (F_{xy} G_{yy} + F_{yy} G_{xy}) \}, \end{aligned}$$

where for the B–Z system (1.3), we have

$$F_x = 1 - k - 2\bar{x} - \frac{k y_c^0}{[\alpha(\mu + \bar{x}) + k]},$$

$$F_{xx} = -2, \qquad F_y = \mu - x, \qquad F_{xy} = -1,$$

$$G_x = -\frac{\alpha k y_c^0}{[\alpha(\mu + \bar{x}) + k]}, \qquad G_y = -[\alpha(\mu + \bar{x}) + k], \qquad G_{xy} = -\alpha,$$

and the remaining derivatives are zeros. Thus, we have (2.14).

(II)

The following computations were done to obtain (2.15).
Using the transformation

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = T^{-1} \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix},$$

where T is given by

$$T = \begin{bmatrix} \omega_0/c & \gamma/c \\ 0 & 1 \end{bmatrix}$$

where

$$c = F_{2x} = \frac{-\alpha k y^0}{[\alpha(\mu + \bar{x}) + k]}$$

$$\gamma = F_{1x} = -F_{2y} = [\alpha(\mu + \bar{x}) + k]$$

and

$$\omega_0 = \sqrt{\det J_0}$$

are all obtained at $y^0 = y_c^0$.

Putting $v = y^0 - y_c^0$, the system (1.2) takes the form

$$\dot{y} = Dy + \psi(y, v),$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \psi(y, v) = \begin{bmatrix} \psi_1(y, v) \\ \psi_2(y, v) \end{bmatrix},$$

D is the diagonal matrix

$$T^{-1}JT = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix},$$

and

$$\psi(y, v) = T^{-1}F(Ty + \bar{x}, y^0 - v).$$

In fact,

$$\begin{bmatrix} \psi_1(y_1, y_2, v) \\ \psi_2(y_1, y_2, v) \end{bmatrix} = \begin{bmatrix} a_1 y_1^2 + a_2 y_1 y_2 + a_3 y_2^2 + a_4 \\ b_1 y_1 y_2 + b_2 y_2^2 + b_3 \end{bmatrix}$$

where

$$a_1 = \frac{\omega_0[\alpha(\mu + \bar{x}) + k]}{\alpha k (v - y^0)}$$

$$a_2 = \frac{[\alpha(\mu + \bar{x}) + k]^2 (2 - \alpha)}{\alpha k (v - y^0)} + 1$$

$$a_3 = \frac{[\alpha(\mu + \bar{x}) + k]^3 (1 - \alpha)}{\omega_0 \alpha k (v - y^0)} - \frac{[\alpha(\mu + \bar{x}) + k]}{\omega_0}$$

$$a_4 = \frac{\alpha k(v-y^0)(\alpha(\mu+\bar{x})+k+1)}{\omega_0[\alpha(\mu+\bar{x})+k]}(\bar{x}^2-(1-k)\,\bar{x})$$

$$b_1 = \frac{\omega_0[\alpha(\mu+\bar{x})+k]}{k(v-y^0)}$$

$$b_2 = \frac{k(v-y^0)}{\omega_0^2}\,b_1^2$$

and

$$b_3 = k(v-y^0) - [\alpha(\mu+\bar{x})+k].$$

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